Limits and Regularity of Graph Sequences

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1 Abstract

A limit of a sequence of graphs is an object that encodes approximate combinatorial information of the sequence. [Lovasz et al citation, 2008] defines such a limit for sequences of dense graph. For example, if a dense graph sequence (G_n) converges to a *graphon* W, then $e_n = \mathsf{e}(G_n)/\mathsf{v}(G_n)^2 \in [0, 1/2)$ (the number of edges of G_n relative to its number of nodes) converges to some e, and e is directly computable from W. As a second example, if M_n denotes the size of the maximum cut of G_n, and $m_n = M_n/\mathsf{v}(G_n)^2 \in [0, 1/2)$ is this size relative to the number of nodes, then m_n converges as $n \to \infty$, and once again this limit is directly computable from W. If (G_n) is any less-than-dense sequence, the limit W is still well-defined, but $W = 0$, which does not carry any information. To account for this, [Benjamini and Schramm citation] have defined a different kind of limit for sparse graph sequences. However, the limit is non-sensical for any greater-than-sparse sequence. This research attempts to fill the gap, by defining limits for a variety of intermediate degree sequences, those strictly between being sparse and being dense.

Given a graph N, does there exist a graph M such that the 1-neighborhoods of every vertex of M are isomorphic to N? When such an M exists, it is called a *mosaic* of N. Bulitko (1977) proved this question to be algorithmically undecidable. We therefore consider a slightly different question: Given a graph N and assuming it has a mosaic, is that mosaic unique, or if not, what characterizes the collection of all such mosaics? A resolution of this question could have applications to sparse regularity lemmas, analogous to Szemeredi's famous regularity lemma for sparse graphs.

1.1 Notation and Definitions

A **simple graph** is a pair (V, E), where V is a finite set of **vertices** or **nodes**, and $E = \{\{x, y\} \subset V : x \neq y\}$ is a set of **edges**. We denote an edge as xy, rather than $\{x, y\}$. If V is ambiguous but G is not, we denote it by $V(G)$, and likewise the edge set by $E(G)$. We denote $\mathsf{v}(G) = |V(G)|$ and $\mathsf{e}(G) = |E(G)|$.

A rooted graph is a triple (V, E, r), where $r \in V$ is a distinguished vertex called the **root**. An **r-neighborhood** $N_r^G(x)$ of simple graph G at root x is the rooted subgraph of G induced by the nodes $\{y \in V : d(x, y) \leq r\}$, where $d(x, y)$ is the length of the shortest path from x to y. (A path from x to y is a sequence of distinct edges starting at x and ending at y.) By this definition, $x \in N_r^G(x)$. If G is clear from context, we prefer $N_r(x)$.

A simple graph M is called a **mosaic** of a 1-neighborhood N if $N_1(x)$ is root-isomorphic to N for all $x \in V$. Synonomously, M is **locally** N (the historical jargon), but we will avoid this terminology.

An *adjacency matrix* of a graph G, with $V = \{1, \ldots, n\}$, is a matrix A_G, where a_{ij}, the ij-th entry of A_G, is 1 if $\{i, j\} \in E$, and 0 otherwise.

The *Erdos-Renyi random graph* $\mathbb{G}(n,p)$, for $n \in \mathbb{N}$ and $p \in (0,1)$, is a probability distrbution over graphs on n vertices. We sample a graph G from this distribution by flipping a biased coin, with probability p landing heads, for each pair of vertices. If the coin comes up heads, connect the pair with an edge, otherwise do not; the decision for each pair is made indepedently from the rest.

A *bipartite graph* is a graph (V,E) such that there exists a partition $V = A \cup B$, where every edge in E has one endpoint in A, and the other in B.

2 Introduction

[Lovasz et al citation, 2008] have defined a very appealing limit for *dense* graph sequences; by definition, a dense graph sequence (G_n) satisfies $\mathsf{e}(G_n) = \Omega(\mathsf{v}(G_n)^2)$. The limit object in this case is called a *graphon*. Graphons, and their surrounding theory, serve primarily as useful theoretical tools that aid in proving theorems; for example, as we will see below, the celebrated Szemeredi Regularity Lemma [citation] follows directly from the compactness of the graphon space, with respect to the cut metric. Besides limits of dense graph sequences, [Benjamini and Schramm citation] have developed a theory of *sparse* graph sequences, which is one where all degrees of all nodes are bounded above by a constant. This theory is far less satisfying than that of the dense case; for example, the limit object, a *graphing*, does not encode connectedness of the sequence, whereas graphons do. An *intermediate degree* sequence is one that is neither sparse nor dense; the sequence of hypercubes provides such an example. Therefore, in this Introduction, I will survey the important results and concepts of the dense case, explain why they are important, and indicate how they may be generalized for the intermediate case.

2.1 The Dense Case

[Lovasz et al] define convergence and limit of sequences of dense graphs in [citation]. They charactertize convergence of graph sequences through three ideas, all of which are equivalent:

1. ...can be precisely quantified as a metric, thereby giving us Cauchy sequences and hope for limit objects, as follows: Given a (large, dense) graph G, let $\sigma_{G,n}$ be a probability distribution over n-node graphs F, where $\sigma_{G,n}(F)$ is the probability that the subgraph induced by n uniformly, distinctly sampled nodes is isomorphic to F. The *variation distance* d_{var} of two such distributions $\sigma_{G,n}, \sigma_{H,n}$ is their maximal difference, $d_{\text{var}}(\sigma_{G,n}, \sigma_{H,n}) = \max_{F:\mathsf{v}(F)=n} |\sigma_{G,n}(F) - \sigma_{H,n}(F)|$. Finally, we can define the sampling distance

$$d_{\text{samp}}(G,H) = \sum_{n=1}^{\infty} 2^{-n} d_{\text{var}}(\sigma_{G,n}, \sigma_{H,n}).$$

   For example, take two Erdos-Renyi random graphs $G \sim \mathbb{G}(n,p), F \sim \mathbb{G}(m,p)$ for large but different m, n. If k nodes are sampled unformly at random each from G, F, then the distributions of the random subgraphs induced by these nodes are exactly the same, and hence have zero variation distance, despite the fact that these two graphs superficially look quite different (different number of nodes, and even if they had the same number, their normalized Hamming distance is $\approx 1/2$ with high probability). Thus, subgraph sampling reflects the fact that two graphs are sampled from the same distribution, which is a very desirable property to have.

2. A sequence of graphs (G_n) converges if, for all graphs H, the sequence $(t(H,G_n))$ converges. The number $t(F,G_n)$ is the *homomorphism density*, defined by $t(F,G_n) = \mathsf{hom}(F,G_n)/\mathsf{v}(G_n)^{\mathsf{v}(F)}$,

where $\mathsf{hom}(F, G_n)$ is the *homomorphism number*, the total number of homomorphisms $F \to G_n$. The normalization $\mathsf{v}(G_n)^{\mathsf{v}(F)}$ is appropriate because there are that many total functions $F \to G_n$, thereby making $t(F, G_n)$ the probability that a random function $F \to G_n$ is a homomorphism. Homomorphisms are intimately connected to subgraph sampling, as the following calculation of $\mathsf{hom}(C_4, G)$ indicates (C_4 is the 4-cycle and G is an arbitrary large graph): C_4 has 3 possible homomorphic images: 1) C_4, 2) P_3, the 3-path, and 3) P_2. The first happens when the homomorphism is injective, the second when one pair of two "opposite" nodes of C_4 are sent to the same node, and the third when both pairs of opposite nodes are sent to two different nodes. We can estimate the numbers of C_4, P_3, and P_2 in G by sampling $4, 3$, and 2 vertices, and taking the induced subgraphs; after this counting, we need to weight by the size of the automorphism group of each homomorphic image to get $\mathsf{hom}(C_4, G)$. This illustrates the very important connecti on between subgraph sampling and homomorphism number.

3. A sequence of graphs (G_n) converges if they are Cauchy in the cut metric $d_\square$. If two labeled graphs G, H with $V(G) = V(H) = \{1, \dots, n\}$ have adjacency matrices A, B, then define

$$\hat{d}_\square(A, B) = \frac{1}{n^2} \max_{S,T \subset V(G)} \left| \sum_{i \in S, j \in T} A_{ij} - B_{ij} \right|.$$

In words, $\hat{d}_\square$ is the maximum difference in edges across a cut of $V(G) = V(H)$, appropriately normalized. But the labeing of the edges of G, H is very arbitrary, so we define the true cut metric $d_\square(G, H) = \min_P \hat{d}_\square(P^{-1}AP, B)$ (P is a permutation matrix), the mimumum difference across all relabelings of G. For the definition of $\delta_\square$ when $\mathsf{v}(G) \neq \mathsf{v}(H)$, we refer the reader to [citation] for details.

More briefly, these ideas are: convergence through random sampling, convergence through counting functions, and convergence through a combinatorial metric. In all cases, though, we define convergence, but never what the limit is. As stated above, the limit is called a graphon, which a symmetric, measurable function $W : [0, 1]^2 \to [0, 1]$ (symmetric meaning $W(x, y) = W(y, x)$). A graphon can be thought of as a continuous generalization of an adjacency matrix.

1. As in point (1) above, one may randomly sample a "subgraph" from a graphon by uniformly sampling some points in $[0, 1]$, and connecting two such points x, y with probability $W(x, y)$, independently from the other pairs, much like with an Erdos-Renyi random graph. This sampling procedure allows us to define the sampling distance $d_{\text{samp}}(U, W)$ between graphons, in the same manner as above.

2. As in (2), if F is a simple graph with $V(F) = \{1, \dots, k\}$, then one may calculate homomorphism densities

$$t(F, W) = \int_{[0,1]^{\mathsf{v}(F)}} \prod_{ij \in E(F)} W(x_i, x_j) dx_1 \cdots dx_k.$$

3. The cut metric, as in (3), also generalizes quite nicely for graphons U, W

$$\hat{d}_\square(U, W) = \sup_{S,T \subset [0,1]} \left| \int_{S \times T} W - U \right|.$$

This discussion would not be complete without pointing out that, much like in the *graph* cut metric, we must minimize over all possible "relabelings" of the graphons U, W to obtain

the true cut metric $d_\square(U, W)$. We "relabel" the set of vertices $[0, 1]$ through a measure-preserving function $\phi : [0, 1] \to [0, 1]$, where $\lambda(\phi^{-1}(E)) = \lambda(E)$ for all measurable $E \subset [0, 1]$, where λ is Lebesgue measure. We define the *measure-preserving transformation* W^ϕ by $W^\phi(x, y) = W(\phi(x), \phi(y))$. The true cut metric $d_\square$ is the minimization over all such relabelings, $d_\square(U, W) = \inf_\phi \hat{d}_\square(U, W^\phi)$. Given two graphons W_1, W_2 if there exist measure-preserving functions ϕ_1, ϕ_2 such that $W_1^{\phi_1} = W_2^{\phi_2}$, we say that W_1, W_2 are *weakly isomorphic*, and identify them. Weak isomorphism plays very nicely with homomorphism density and the cut metric, in that $t(F, W_1) = t(F, W_2)$ for all simple graphs F, iff W_1, W_2 are weakly isomorphic, iff $d_\square(U, W) = 0$. (We need two such measure-preserving functions to allow for transitivity and symmetry of the equivalence relation.)

If $\mathsf{e}(G_n) < \Omega(\mathsf{v}(G_n))$, then with high probability, the subgraph of G_n induced by some uniformly random chosen nodes will be empty, for sufficiently large n. Therefore, every less-than-dense graph sequence converges in this sense, but the limit will not reveal any information of the sequence.

2.1.1 Quasirandomness and Regularity

There is a large conceptual leap going from a the idea of a probability distribution to a random sample from a distribution. In the traditional approach to probability through measure theory, we study distributions, but we do not ever talk about what it means to sample from them. How can we say that a pseudo-random number generator, returning digits in the range $[0, 9]$, is "sufficiently random"? What does this mean? How can deterministic objects and processes be considered random-like?

Similarly, how can we consider a given, deterministic graph as "random-like", or pseudorandom? We can answer this question for bipartite graphs $G = (V, E)$, with partition $V = A \cup B$, through the idea of ϵ-regularity. Let X, Y be disjoint subsets of V. Define $d(X, Y) = |E(X, Y)|/(|X||Y|)$ as the *edge density* of this pair (X, Y), where $E(X, Y) \subset E$ are those edges with one endpoint in X, and the other in Y. Then G is considered ϵ-pseudorandom if for all subsets $X \subset A,\ Y \subset B$ with $|X| \geq \epsilon|A|,\ |Y| \geq \epsilon|B|$, there is $|d(X, Y) - d(A, B)| < \epsilon$. Informally, G is ϵ-pseudorandom if for all subsets of comparable size of A, B the edge density between these subsets is sufficiently close to the edge density between the partition $V = A \cup B$.

Why is ϵ-psuedorandom considered a good idea for pseudorandomness? Consider a slight variant of the Erdos-Renyi random graph, $\mathbb{B}(n, m, p)$, a distribution over bipartite graphs with parts A, B of size $|A| = n,\ |B| = m$. For each pair of nodes, with one node in A and the other node in the B, we flip a biased coin, with probability p for heads. If the coin is heads for the pair, we put an edge between the nodes, otherwise we put no edge; the decisions for all pairs are made independently. For all subsets $X \subset A,\ Y \subset B$, the expectation of $E(X, Y)$ is $\frac{1}{2}p|X||Y|$, with variance $\frac{1}{2}p(1-p)|X||Y|$. So $d(X, Y) = d(A, B) = \frac{1}{2}p$, and thus $|d(X, Y) - d(A, B)| = 0$ in expectation.

2.2 The Sparse Case

In this section, we will discuss the motivation for the problem posed in the abstract. A regularity lemma is a result that decomposes a sufficiently large graph into another object of bounded size that approximates the large graph in some manner. The original regularity lemma is that of Szemeredi. His result is trivial for graphs of bounded degree, namely because the perimeter of a subset of vertices is small compared to the number of vertices. The following regularity lemma, due to Alon (unpublished), however, is nontrivial for graphs of bounded degree.

Theorem (Alon Sparse Regularity Lemma)**.** *For every $d, r \geq 1$ and $\epsilon > 0$ there is a $k = k(d, r, \epsilon)$ such that for every graph G with degrees bounded by d there is a graph H with degrees bounded by*

d and $|V(H)| \leq k$, such that the distributions r-neighborhoods in G and H are closer than ϵ in variation distance.

A "distribution of r-neighborhoods" is the probability distribution σ_r^G, defined over r-neighborhoods as follows: $\sigma_r^G(N) = |\{x \in V(G) : N_r(x) \text{ is root-isomorphic to } N\}|/\mathsf{v}(G)$. If G is clear from contet, we prefer to denote σ_r. The "variation distance" is the metric $d_{\text{var}}(G,H) = \sup_N |\sigma_r^G(N) - \sigma_r^H(N)|$.

Proof Fix d and r. Let $\mathcal{N}$ be the collection of all possible r-neighborhoods of degrees bounded by d. First, $\mathcal{N}$ is finite. Indeed, $N \in \mathcal{N}$ has at most $\sum_{n=0}^{r} d^n < d^{r+1}$ vertices, so $|\mathcal{N}| < 2^{\binom{d^{r+1}}{2}}$. Choose $\mathcal{H} \subset \{H : \max_v \deg_H(v) \leq d\}$ so that for all $H_1, H_2 \in \mathcal{H}$ there is $d_{\text{var}}(H_1, H_2) \geq \epsilon$, and so that $\mathcal{H}$ is maximal. By definition, any graph G with degrees bounded by d will be within ϵ in variation distance to some $H \in \mathcal{H}$. Each $H \in \mathcal{H}$ corresponds with an (open) cube C_H of volume $\epsilon^{|\mathcal{N}|}$. The cubes $\{C_H : H \in \mathcal{H}\}$ are pairwise disjoint, so $|\mathcal{H}|\epsilon^{|\mathcal{N}|} \leq 1$, implying $|\mathcal{H}| \leq (1/\epsilon)^{|\mathcal{N}|}$, so $\mathcal{H}$ is finite. Therefore take $k(d,r,\epsilon) = \max_{H \in \mathcal{H}} \mathsf{v}(H)$. □

The proof of the Alon Sparse Regularity Lemma does not construct H, nor does it give an obvious bound on k. Our aim is to do both of these things, and this is what motivates the problems posed in the abstract. Our approach has been as follows: given an r-neighborhood N, construct a connected graph M such that $\sigma_r^M(N)$ is maximal over all M (or within some δ of the supremum if no such maximum exists). We then combine these constructed M by taking disjoint unions of an appropriate number of copies of them to get H.

To clarify the last sentence above, we will construct H for the special case where $r = 1$, and where all 1-neighborhoods N with $\sigma_1(N) > 0$ have mosaics, assuming we are given these mosaics. Let $\mathcal{N}$ be those 1-neighborhoods with degrees bounded by d that have mosaics. Let $\text{vm}(N)$ be the fewest number of vertices of any mosaic of $N \in \mathcal{N}$, and denote such a mosaic M_N. Let $L = \text{lcm}_{N \in \mathcal{N}} \text{vm}(N)$, and $L_G = \text{lcm}\{\text{vm}(N) : \sigma_1(N) > 0\}$. Choose n so that $(2^n/(d+1)^2 + 1)^{-1} + 2^{-n} < \epsilon$. Then

$$H = \bigsqcup_{N : \sigma_1(N) > 0} M_N^{\lceil \sigma_1(N) 2^n \rceil L_G / \text{vm}(N)},$$

where M_N^m denotes m disjoint copies of M_N. Clearly $\mathsf{v}(H) \leq L_G(2^n + 2^{(d+1)^2}) \leq L(2^n + 2^{(d+1)^2})$, so we may take $k = L(2^n + 2^{(d+1)^2})$. We leave it to the reader to verify that $|\sigma_1^G(N) - \sigma_1^H(N)| < \epsilon$ for all $N \in \mathcal{N}$.

2.3 Mosaic Existence

Historically, the question of mosaic existence is known as the Zykov-Trahtenbrot problem. In 1977, Bulitko proved that no algorithm can answer this question in finite time. It is hopeless then to compute or put a bound on L. We can only prove existence or nonexistence of mosaics of *particular* 1-neighborhoods. We can also prove properties of mosaics. This is the line of inquiry that this paper follows. The first result along this line is the following.

Proposition.

Along a related line of inquiry, we aim to characterize those graphs that have unique mosaics. For some examples, the simple graph P_4 (the path on 4 vertices) is the graph with the fewest number of edges that has more than one mosaic; see figure (1) for a couple example constructions. On the other hand, every simple complete graph K_n has a unique mosaic, namely K_{n+1}. So too do the graphs C_3, C_4, and C_5; see figure (2) for the mosaics. The uniqueness of C_3 is trivial, that of C_4 is not as trivial but still easy to see, and C_5 is much harder. The mosaic of C_6 is not unique. Therefore, I

Conjecture. *The mosaic of C_n, the n-cycle, exists, and is not unique for $n \geq 6$.*

We will not talk about uniqueness past this point, only existence. In this direction, we state and prove a proposition. This proposition intuitively says that we only need to consider existence of mosaics for 1-neighborhoods that are flank-connected.

Proposition. *Given a 1-neighborhood N, if the connected components $\{A_i\}_{i=1}^n$ of N^F all have mosaics, then N has a mosaic. Moreover, given mosaics $\{M_i\}_{i=1}^n$ of the $\{A_i\}$, we can construct a mosaic M of N.*

Proof We prove the proposition for $n = 2$. The proof below makes no use of the fact that A_1, A_2 are connected, so the general case $n > 2$ follows by induction.

In order to give an intuition for the proof of $n = 2$, we first do a graphic example. Assume that $N, A_1, A_2, \dot{A}_1, \dot{A}_2, M_1, M_2, M$ are as illustrated in figure (3) of the attachment. In the M illustration, notice that there are three M_1 placed in a triangular arrangement, each connected with a large number of edges. The triangular arrangement comes from the fact that M_2 is a triangle. The M_1 drawings are linear translations of each other. Looking closer at the connections between two M_1, see that there are six edges total, connecting each node with its translated counterpart.

Now a formal proof of the case $n = 2$. Arbitrarily label the vertices of M_1 with distinct labels. Make $|V(M_2)|$ copies of M_1. There are $|V(M_2)|$ vertices with the same label, and they (currently) form an independent set. So connect vertices with the same label by a copy of M_2. The resulting graph is M, as desired. □

We will soon give conditions for the existence of mosaics of certain kinds of flank-connected 1-neighborhoods. But first we must talk about words.

Definition. *If two nodes $x, y \in V(G)$ satisfy $N_1(x) = N_1(y)$, then x, y are called **familiar**. Clearly familiarity is an equivalence relation. The equivalence classes of $V(G)$ with respect to familiarity are called **families**.*

Proposition. *The families $\{F_i\}$ of G are cliques. Moreover, for all pairs of families F_i, F_j, either $F_i \cup F_j$ is a clique, or F_i and F_j have no connections between them.*

Proof The first assertion is obvious. For the second assertion, either there exists an edge $xy \in E(F_i, F_j)$, or there does not. For the former possibility, it follows that we have an edge $wy \in E(F_i, F_j)$ for all $w \in F_i$, by definition of familiarity. Therefore we have that $wz \in E(F_i, F_j)$ for all $w \in F_i, z \in F_j$; in other words, $F_i \cup F_j$ is a clique. □

In light of the regular structure between families, we may consider a kind of quotient graph with respect to familiarity.

Definition. *The **family-graph** of G is a weighted graph $\mathcal{F}(G)$ where families $F \subset V(G)$ correspond with nodes $x_F \in V(\mathcal{F}(G))$, and two nodes $x_F, x_{F'}$ are connected iff there exists a from F to F'. The weight of x_F is $|F|$.*

Proposition. *$\mathcal{F}(\mathcal{F}(G)) = \mathcal{F}(G)$, ignoring weights. All the weights of $\mathcal{F}(\mathcal{F}(G))$ are one.*

Proof Fix $x_F, x_{F'} \in V(\mathcal{F}(G))$ such that $N_1(x_F) = N_1(x_{F'})$. Then the nodes $u \in F, v \in F'$ satisfy $N_1(u) = N_1(v)$, implying that $F = F'$, and hence $x_F = x_{F'}$. □

We are nearly ready to state and prove the theorem, only needing a few more definitions.

Theorem. *Let (N, r) be a 1-neighborhood. Then N has a mosaic M if (1) all nodes of $\mathcal{F}(N)$ have equal weight, and (2) $\mathcal{F}(N - R)$ has a mosaic $\mathcal{M}$, where R are the set of vertices of N familiar to the root. Conversely, if the root of $\mathcal{F}(N)$ has a higher weight than any other node, then N does not have a mosaic.*

Proof

For the first part, just take $M = \mathcal{M}(n)$, the n-blowup of $\mathcal{M}$, where n is the weight of the root node of $F(N)$.

The second part follows from the fact that if two nodes are familiar in a neighborhood of either of those points, then they are familiar in the neighborhood of any of their neighbors. $\square$

This theorem proves the existence (and non-existence) of mosaics for a huge class of flank-connected 1-neighborhoods. For example, there is exactly one 1-neighborhood with $d \leq 3$ that does not have a mosaic, namely the graph in figure (4). This 1-neighborhood is indeed a double-neighborhood, and does not satisfy the hypotheses of the theorem. On the other hand, a few small-size graphs are not double-neighborhoods, and do not have mosaics. The smallest such graph I could find is in figure (5). This is easily my favorite graph I have come across so far, both because of its curious nature, and its visual beauty.

Over my next research phase, I intend first to characterize why the graph of figure (5) does not have a mosaic, and second to prove that the cycles have unique mosaics.

Theorem: Let N be a 1-neighborhood, and fix $d \geq \deg N$. Let $p = \sup\{\sigma_1^G(N) : \deg G \leq d\}$. If $p < 1$, then there exists a graph G such that $\deg G \leq d$ and $\sigma_1^G(N) = p$. In other words, the supremum in the definition of p is in fact a maximum when $p < 1$.

Proof: First we define notation. For a graph G, Let $I(G)$ be those $x \in V(G)$ such that $N_1(x)^F$ is isomorphic to an induced subgraph of N^F. Let $H(G) = \{x \in V(G) : N_1(x) \approx N\}$. ($H$ because these nodes are happy with their neighborhoods.) Let $U(G) = V(G) - H(G)$. Let

$$\mathcal{G}_v = \{G : \deg G \leq d,\ \mathsf{v}(G) \leq v,\ \text{and } i(G) \geq |H(G)| + 1\}.$$

Let

$$p_v = \max\{\sigma_1^G(N) : G \in \mathcal{G}_v\}.$$

Assume towards a contradiction that $\sigma_1^G(N) < p$ for all simple graphs G with $\deg G \leq d$. Call this assumption the "main assumption", because we will be making lots of assumptions towards contradictions.

We claim that $p_v \to p$. Indeed, if a graph G has $\deg G \leq d$, we could tack on a degree 1 node a to $b \in V(G)$ with $\deg b < d$ (call the resulting graph G'), giving $\sigma_1^{G'}(N) = |H(G)|/(\mathsf{v}(G)+1)$ and $G' \in \mathcal{G}_{\mathsf{v}(G')}$. If b does not exist, we create it by arbitrarily deleting an edge between one node b with $\deg b = d$, and some other node. Call the resulting graph G''. Now tack on a to b, and call the result G'. In this case, we have $\sigma_1^G(N) \geq \sigma_1^{G'}(N) \geq \frac{|H(G)|-(d+1)}{\mathsf{v}(G)+1}$. In either case, $\sigma_1^G(N) \sim \sigma_1^{G'}(N)$ as $\mathsf{v}(G) \to \infty$. By our main assumption indeed $\mathsf{v}(G) \to \infty$, so this proves the claim $p_v \to p$.

For $n \geq 2$, let $v_n = \min\{v : p_v > (1-1/n)p\}$, and choose a graph $G_n \in \mathcal{G}_{v_n}$ with $\sigma_1^{G_n}(N) = p_{v_n}$.

There exists m such that there exists $S \subset U(G_m)$ with $|S| = \mathsf{v}(N)$, with $\min_{x \neq y \in S} d(x,y) \geq 4$, and with $I(G_m) \cap S \neq \emptyset$. (In fact, we can make both $|S|$ and the minimum distance arbitrarily large, but that's unnecessary.) Indeed, for any $T \subset U(G_n)$, we have

$$|\{x \in V(G_n) : \exists y \in T \text{ s.t. } d(x,y) \leq 3\}| \leq (1 + d + d^2 + d^3)|T| < d^4|T|.$$

Notice that $p < 1$ implies $|U(G_m)| \to \infty$, so we can choose m so that $|U(G_m)| \geq d^4 \sum_{i=1}^{\mathsf{v}(N)} i = d^4 \binom{\mathsf{v}(N)+1}{2}$. Therefore S exists as defined above, because we may iteratively choose points in $U(G_m)$ lying outside distance 3 of the previously chosen points, starting with a point in $I(G_m) \cap U(G_m)$.

We can use this set S to construct a graph G' such that $\sigma_1^{G'}(N) > \sigma_1^{G_m}(N)$ and $G' \in \mathcal{G}_m$. This contradicts the definition of p_{v_m}. Indeed, fix a $u \in I(G_m) \cap S$. Then simply connect nodes of $N_1(u)$ to nodes of S, and nodes of S to other nodes of S, so that $N_1(x) \approx N$. Call the new graph G'.

Fix $x \in H(G_m)$, and let $x' \in V(G')$ be its corresponding node. We claim $x' \in H(G')$. Indeed, for two $y, z \in V(N_1^{G_m}(x))$, then $y'z' \in E(G')$ only if:

1. $yz \in E(G_m)$; or
2. $y, z \in S$; or
3. $yu \in E(G_m)$ and $z \in S$; or
4. $zu \in E(G_m)$ and $y \in S$.

Possibility (2) contradicts $d(y, z) \leq 2$. Possibility (3) implies $d(z, u) = 3$, contradicting $z, u \in S$; analogously, (4) implies a contradiction. Therefore, $yz \in E(G)$, and our claim $x' \in H(G')$ is proven. Conclude from this that $\sigma_1^{G'}(N) > \sigma_1^{G_m}(N)$, and $|I(G_m)|$

Of course, we must show that u exists. For the sequence $\{G_n\}$ described above, there is no guarantee. So we must massage the construction to force a u.

3 References

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